IMM 368
June 1968

Courant Institute of
Mathematical Sciences

Some Generalized Eigenfunction Expansions and Uniqueness Theorems

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Prepared under Contract Nonr-285(55)
with the Office of Naval Research NR 062-160

New York University

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SOME GENERALIZED EIGENFUNCTION EXPANSIONS AND UNIQUENESS THEOREMS

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This report represents results obtained at the Courant Institute of Mathematical Sciences, New York University, with the Office of Naval Research, Contract Nonr-285(55).

Abstract

A generalized eigenfunction expansion method, Churchill's method, and a transform method are used to investigate the uniqueness of the solution of the equation

$$\frac{\partial}{\partial y}\, p(y)\, \frac{\partial}{\partial y}\, \phi(x,y) + q(y)\phi + r(y)\, \frac{\partial^2 \phi}{\partial x^2} = 0 \,, \qquad -\infty < x < \infty \,, \quad 0 < y < 1$$

subject to the boundary conditions

$$\phi_y(x,0) + \alpha_o \phi_{xx}(x,0) + \beta_o \phi(x,0) = 0$$

and

$$\phi_y(x,1) + \alpha_1 \phi_{xx}(x,1) + \beta_1 \phi(x,1) = 0 \,.$$

1. Introduction

A. Weinstein [1] showed that if $\phi(x,y)$ is a potential function which is required to satisfy

$$(1.1) \qquad \phi_{xx}(x,y) + \phi_{yy}(x,y) = 0 , \qquad -\infty < x < \infty , \quad 0 < y < 1 ,$$

$$(1.2) \qquad \phi_y(x,0) = 0 ,$$

$$(1.3) \qquad \phi_y(x,1) = p\phi(x,1) , \qquad p > 0 ,$$

and if λ^* is the unique positive root of

$$(1.4) \qquad \sqrt{\lambda^*} \tanh \sqrt{\lambda^*} = p$$

where p is a constant, then

$$(1.5) \qquad \phi(x,y) = [A_o \cos x \sqrt{\lambda^*} + B_o \sin x \sqrt{\lambda^*}] \cosh y \sqrt{\lambda^*}$$

is the only bounded function which satisfied the above conditions. Weinstein's proof of this is based on a completeness theorem connected with an eigenvalue problem which he introduces in the following way. Let $\phi_{xx}(x,y)$ in (1.1) be replaced by $-\lambda\psi(y)$ while ϕ_{yy}, ϕ_y and ϕ in (1.1)-(1.3) are respectively replaced by ψ_{yy}, ψ_y and ψ. These substitutions impose the conditions

$$(1.6) \qquad \psi_{yy}(y) - \lambda\psi(y) = 0 , \qquad 0 < y < 1 ,$$

$$(1.7) \qquad \psi_y(0) = 0 ,$$

$$(1.8) \qquad \psi_y(1) = p\psi(1) , \qquad p > 0 ,$$

and these conditions define a standard Sturm-Liouville eigenvalue problem. The eigenvalues are the numbers which satisfy

$$(1.9) \qquad \sqrt{\lambda} \tanh \sqrt{\lambda} = p , \qquad p > 0 .$$

The eigenvalues are all real and we can consider them as ordered with respect to absolute magnitude. The sole positive eigenvalue will be denoted by λ^*. The eigenfunctions are given by

$$(1.10) \qquad \psi_n(y) = \mu_n \cosh y \sqrt{\lambda_n} , \qquad n = 0,1,2\ldots ,$$

and they satisfy the orthogonality condition

$$\int_0^1 \psi_n(y)\, \psi_m(y) dy = 0 , \qquad m \neq n .$$

It is well known that the set (1.10) is complete. This implies that the function $\phi(x,y)$ can be uniquely expressed in the form

$$\phi(x,y) = \sum_{n=0}^{\infty} \frac{h_n \psi_n(y)}{\int_0^1 \psi_n^2(y) dy} .$$

In this expansion the coefficient $h_n(x)$ is

$$h_n(x) = \int_0^1 \phi(x,y)\psi_n(y)dy$$

and it must satisfy

$$h_n''(x) = \int_0^1 \phi_{xx}(x,y)\psi_n(y)dy = -\int_0^1 \phi_{yy}(x,y)\psi_n(y)dy$$

$$= -\left[\left|\phi_y\psi_n - \phi\psi_{ny}\right|_0^1 + \int_0^1 \phi(x,y)\psi_{yy}(x,y)dy\right]$$

$$= -\lambda_n h_n(x) \ .$$

That is, we must have

$$h_n(x) = a_n \cos x\sqrt{\lambda_n} + b_n \sin x\sqrt{\lambda_n} \ .$$

Hence if $\phi(x,y)$ is to be bounded in the strip, $0 \leq y \leq 1$, $-\infty < x < \infty$, then the only non zero coefficient, h_n, that can be admitted is

$$h_n = h^* = a^* \cos x\sqrt{\lambda^*} + b^* \sin x\sqrt{\lambda^*}$$

and this leads to the result (1.5).

Let us turn now to the equation

$$\phi_{xx}(x,y) + \phi_{yy}(x,y) = 0 \ , \qquad -\infty < x < \infty \ , \quad 0 < y < 1 \ , \tag{1.11}$$

subject to the boundary conditions

(1.12) $$\phi_y(x,0) = 0 ,$$

and

(1.13) $$\phi_y(x,1) + \alpha\phi_{xx}(x,1) = 0 , \qquad \alpha > 0 ,$$

where α is a positive constant. If we attempt to analyze the solutions of this system by following Weinstein's eigenfunction method we are led to the eigenvalue problem defined by

(1.14) $$\psi_{yy}(y) - \lambda\psi(y) = 0 , \qquad 0 < y < 1 ,$$

(1.15) $$\psi_y(0) = 0 ,$$

(1.16) $$\psi_y(1) = \alpha\lambda\psi(1) .$$

This is not a standard Sturm-Liouville problem because the eigenparameter appears in the boundary condition (1.16). The implications of this are substantially different from those of the eigenvalue problem discussed above. In order to see this, take $\alpha = 1$. For this value the eigenvalues must satisfy

$$\tanh \sqrt{\lambda} = \sqrt{\lambda} .$$

Except for $\lambda = \lambda_o = 0$ the eigenvalues are negative and we consider them to be ordered with respect to absolute magnitude. The eigenfunctions $\{\psi_n(y)\}$ are

(1.17) $$\psi_n(y) = \mu_n \cosh y\sqrt{\lambda_n} , \qquad n = 0,1,2\ldots ;$$

and they must satisfy the generalized orthogonality condition

$$(1.18) \qquad \psi_n(1)\psi_m(1) - \int_0^1 \psi_n(y)\psi_m(y)dy = 0 , \qquad m \neq n .$$

Now if we wish to proceed in accordance with Weinstein's method we need to know whether or not an arbitrary twice differentiable function f(y) can be expressed in the form

$$(1.19) \qquad f(y) = \sum_{n=0}^{\infty} k_n\psi_n(y) = \ell + \sum_{n=1}^{\infty} \ell_n\psi_n(y)$$

where the ψ_n's are given by (1.17). If the representation (1.19) is valid with the series uniformly convergent for $0 \leq y \leq 1$, then the coefficients ℓ_n are fixed by

$$(1.20) \qquad \ell_n = \frac{2}{\mu_n^2 \sinh^2 \sqrt{\lambda_n}} \left[\psi_n(1)f(1) - \int_0^1 f(y)\psi_n(y)dy\right] .$$

This suggests the tentative association of f(y) with the series determined by (1.20), that is,

$$(1.21) \quad f(y) \sim \ell + 2\sum_{n=1}^{\infty} \frac{\left[\psi_n(1)f(1) - \int_0^1 f(y)\psi_n(y)dy\right]}{\mu_n^2 \sinh^2 \sqrt{\lambda_n}} \psi_n(y) .$$

This association, however, must be rejected because if we choose $f(y) = y^2$ and note that

$$\psi_n(1) - \int_0^1 y^2\psi_n(y)dy = 0 , \qquad n \geq 1 ,$$

we see that (1.21) forces us to associate y^2 with the constant ℓ. This shows that the set of eigenfunctions (1.17) is inadequate for expansion purposes. We therefore conclude from the foregoing remarks that the applicability of Weinstein's method to the case in hand depends on finding a set of functions $\{\chi_n(y)\}$ which contains $\{\psi_n(y)\}$ and admits the expansion

$$f(y) = \sum_{n=0}^{\infty} c_n \chi_n(y) .$$

The major part of this report is concerned with the development of expansions which allow an extension of Weinstein's method. In Section 2, using a Poincaré-Birkhoff formulation, we show how the eigenfunction method can be applied to the equation

$$(1.22) \quad \frac{\partial}{\partial y} p(y) \frac{\partial}{\partial y} \phi(x,y) + q(y)\phi + r(y) \frac{\partial^2 \phi}{\partial x^2} = 0 , \qquad -\infty < x < \infty , \quad 0 < y < 1 ,$$

subject to the boundary conditions

$$(1.23) \qquad \phi_y(x,0) + \alpha_0 \phi_{xx}(x,0) + \beta_0 \phi(x,0) = 0 ;$$

and

$$(1.24) \qquad \phi_y(x,1) + \alpha_1 \phi_{xx}(x,1) + \beta_1 \phi(x,1) = 0 .$$

Various analyses of the system (1.22)-(1.24) appear in the literature but as far as the author knows these analyses depend on imposing restrictions on the real α's and β's. One of the features of the development below is that these constants are left unrestricted.

Partial differential systems of the above type arise of course in connection with uniqueness theorems for the associated non-homogeneous equations. In many applications, however, they also arise in just the above homogeneous form. For example, the the system (1.22)-(1.24) emerges for consideration in hydrodynamics in connection with flows in a horizontal channel with rectangular cross section. A basic problem is to discover relations between certain parameters which insure a parallel flow. Problems of this type led to the analysis contained in this report. An example is discussed in Section 2.

Sections 3 and 4 are devoted to the analysis of (1.22)-(1.24) by methods related to, but different from the method of Section 2. Section 3 contains an extension of a method devised by R. Churchill whereby, under certain circumstances, an eigenvalue problem with boundary conditions dependent on the eigenparameter can be reduced to a standard problem. Section 4 contains a discussion of the use of a transform method for the study of uniqueness questions about (1.22)-(1.24).

2. A Generalized Eigenfunction Expansion and the Extension of Weinstein's Method

The equation

$$(2.1)\qquad \frac{\partial}{\partial y}\, p(y)\, \frac{\partial}{\partial y}\, \phi(x,y) + q(y)\phi + r(y)\, \frac{\partial^2 \phi}{\partial x^2} = 0\ , \qquad -\infty < x < \infty\ , \quad 0 < y < 1\ ,$$

subject to the boundary conditions

$$\phi_y(x,0) + \alpha_o \phi_{xx}(x,0) + \beta_o \phi(x,0) = 0 \; ; \tag{2.2}$$

$$\phi_y(x,1) + \alpha_1 \phi_{xx}(x,1) + \beta_1 \phi(x,1) = 0 \; , \tag{2.3}$$

possesses the trivial solution $\phi \equiv 0$. Are there any other solutions which are bounded in the strip $-\infty < x < \infty$; $0 \leq y \leq 1$? We assume that p, q and r are real functions of y such that $p(y) > 0$ and $r(y) > 0$ for $0 \leq y \leq 1$. We also assume that the α's and β's are real but otherwise unrestricted.

In order to analyze (2.1)-(2.3) let us introduce the eigenvalue problem defined by the equations

$$\frac{d}{dy}\, p(y)\, \frac{d}{dy}\, \psi(y) + q(y)\psi - \lambda r(y)\psi = 0 \; , \qquad 0 < y < 1 \; , \tag{2.4}$$

$$\psi_y(0) + \beta_o \psi(0) = \alpha_o \lambda \psi(0) \; ; \tag{2.5}$$

$$\psi_y(1) + \beta_1 \psi(1) = \alpha_1 \lambda \psi(1) \; . \tag{2.6}$$

This problem can be regarded as one which is suggested by the application of the method of separation of variables to (2.1)-(2.3).

The eigenfunction $\psi_n(y) \not\equiv 0$ is a non-trivial solution of (2.4)-(2.6) which corresponds to the eigenvalue $\lambda = \lambda_n$. Such a function is unique to within a multiplicative factor. These eigenfunctions constitute a denumerable set $\{\psi_n(y)\}$ such that

$$(2.7) \quad \int_0^1 r(y)\psi_n(y)\psi_m(y)dy - p(1)\alpha_1\psi_n(1)\psi_m(1) + p(0)\alpha_o\psi_n(0)\psi_m(0) = 0 \; , \qquad m \neq n \; .$$

The denumerable set of eigenvalues $\{\lambda_n\}$ has the point at infinity as its only limit point; and we take the eigenvalues as ordered in such a way that

$$|\lambda_o| \leq |\lambda_1| \leq |\lambda_2| \cdots \leq |\lambda_n| \leq \cdots \; .$$

The sequence of operations defined by the left-hand side of (2.7) occurs so frequently in the sequel that it is convenient to use a special symbol for it. We define $Q[\Omega,\chi]$ by

$$(2.8) \qquad Q[\Omega(y),\chi(y)] = \begin{aligned} &\int_0^1 r(y)\Omega(y)\chi(y)dy \\ &- p(1)\alpha_1\Omega(1)\chi(1) \\ &+ p(0)\alpha_o\Omega(0)\chi(0) \; . \end{aligned}$$

We are interested here in the problem of representing a twice differentiable function f(y) in the form

$$f(y) = \sum_{n=0}^{\infty} c_n\chi_n(y)$$

where the set $\{\chi_n(y)\}$ contains the set $\{\psi_n(y)\}$. A basic expansion formula can be obtained from a study of the Green's functions associated with (2.4)-(2.6). This function $G(y,\eta,\lambda)$ is defined by the equation

$$(2.9) \qquad LG(y,\eta,\lambda) - \lambda r(y)G = \delta(y-\eta) \ , \qquad 0 < y,\eta < 1$$

subject to the boundary conditions

$$(2.10) \qquad G_y(0,\eta,\lambda) + \beta_0 G(0,\eta,\lambda) = \alpha_0 \lambda G(0,\eta,\lambda) \ ;$$

and

$$(2.11) \qquad G_y(1,\eta,\lambda) + \beta_1 G(1,\eta,\lambda) = \alpha_1 \lambda G(1,\eta,\lambda) \ .$$

The symbol L is used to denote the operation defined by

$$(2.12) \qquad L\Omega \equiv \frac{d}{dy}\, p(y)\, \frac{d}{dy}\, \Omega(y) + q(y)\Omega(y) \ ;$$

and with respect to this we have the fundamental and indispensable identity

$$(2.13) \qquad \int_\eta^t \chi L\Omega dy = \Big| p(\chi\Omega_y - \chi_y\Omega) \Big|_\eta^t + \int_\eta^t \Omega L\chi dy \ .$$

The symbol $\delta(y-\eta)$ on the right-hand side of (2.9) denotes the generalized function such that when it is multiplied by a piecewise continuous function $F(y)$ (with a finite number of finite jumps) and integrated, the result means

$$\int_a^b \delta(y-\eta)F(y)dy = \frac{1}{2}\,[F(\eta+0) + F(\eta-0)]$$

when $a < \eta < b$.

The function $G(y,\eta,\lambda)$ can be expressed in the form

$$G(y,\eta,\lambda) = \frac{\theta(y,\eta,\lambda)}{\omega(\lambda)}$$

where each of $\theta(u,\eta,\lambda)$ and $\omega(\lambda)$ is an entire function of $\lambda = z$ regarded as a complex variable. The zeros of $\omega(\lambda)$ are just the eigenvalues of (2.4)-(2.6). A complex variable method depending on the use of the Cauchy integral formula coupled with the theory of residues, can be used to show that if C is a circle of radius ρ centered at the origin of the complex (λ=z)-plane and containing the first m+1 eigenvalues λ_n, n = 0,1,2,...,m, then

$$(2.14) \qquad \lim_{\rho \to \infty} \frac{1}{2\pi i} \oint_C \frac{G(y,\eta,z)}{z}\, dz$$

is a null function while

$$(2.15) \qquad \lim_{\rho \to \infty} \frac{1}{2\pi i} \oint_C \frac{G(0,\eta,z)}{z}\, dz = 0$$

and

$$(2.16) \qquad \lim_{\rho \to \infty} \frac{1}{2\pi i} \oint_C \frac{G(1,\eta,z)}{z}\, dz = 0 \; .$$

These results imply that if F(y) is such that $\int_0^1 F^2(y)dy$ exists, then

$$(2.17) \qquad \lim_{\rho \to \infty} \frac{1}{2\pi i} \oint_C \frac{\int_0^1 F(y)G(y,\eta,z)}{z}\, dy = 0 \; .$$

Also if f(y) is a twice differentiable function such that

$$Lf(y) = F(y)$$

then

$$(2.18) \qquad f(\eta) = - \lim_{\rho \to \infty} \frac{1}{2\pi i} \oint_C Q[G(y,\eta,z),f(y)]dz \, .$$

The formula (2.18) is called the Poincaré-Birkhoff formula. It was noted in a less general form by Poincaré [2] during some work on a special problem in partial differential equations. Then Birkhoff [3] proved the formula for an ordinary n^{th} order boundary value problem subject to certain regularity assumptions and with the eigenparameter absent from the boundary conditions. Later, Tamarkin [4] showed that the formula is valid for a wide class of boundary value problems with the parameter present in the boundary conditions. Since then, the formula has been proved by Wilder [5], Langer [6], Rasulov [7] and others under less restrictive conditions than those used by Tamarkin. The proofs of (2.18), as given by the authors noted above, depend upon explicit asymptotic estimates of the behavior of eigenfunctions and eigenvalues as $\lambda \to \infty$. For a proof of (2.18) with respect to the second order system (2.9)-(2.11); and one which does not depend on specific asymptotic evaluations, see Peters [8].

The formula (2.18) leads to the expansion of $f(y)$ into an infinite sum of residues. If C_n is a circle with center at λ_n containing no other eigenvalue we have

$$(2.19) \qquad f(y) = -\sum_{n=0}^{\infty} \frac{1}{2\pi i} \oint_{C_n} Q[G(t,y,z),f(t)]dz$$

$$= \sum_{n=0}^{\infty} Q\left[\frac{1}{2\pi i} \oint_{C_n} G(t,y,z)dz, f(t)\right] .$$

If $\omega(z)$ has a zero of order k at $z = \lambda_n$, the corresponding term in the expansion (2.19) is

$$(2.20) \qquad Q[\tilde{\theta}_1(t,y,n),f(t)]$$

where $\tilde{\theta}_1(t,y,n)/(z - \lambda_n)$ comes from the Laurent expansion of $G(t,y,z)$ for the neighborhood of $z = \lambda_n$. The function $\tilde{\theta}_1$ can be obtained by substituting the expansion

$$G(t,y,z) = \sum_{\ell=-k}^{0} (z-\lambda_n)^{\ell}\, \tilde{\theta}_{-\ell}(t,y,n) + \sum_{\ell=1}^{\infty} (z-\lambda_n)^{\ell}\, \theta_{\ell}(t,y,n)$$

in the equation

$$LG(t,y,z) - \lambda_n r(t)G - (z-\lambda_n)rG = \delta(t-y)$$

and the boundary conditions

$$G_t(0,y,z) + \beta_o G(0,y,z) = \alpha_o \lambda_n G(0,y,z) + \alpha_o (z-\lambda_n) G(0,y,z) ,$$

$$G_t(1,y,z) + \beta_1 G(1,y,z) = \alpha_1 \lambda_n G(1,y,z) + \alpha_1 (z-\lambda_n) G(1,y,z) ;$$

which define $G(t,y,z)$. We find from these equations, after equating coefficients of like powers of $(z-\lambda_n)$, that the circumflexed quantities must satisfy

$$(2.21) \qquad \tilde{\theta}_k(t,y,n) = \gamma_n \psi_n(t) ,$$

$$(2.22) \qquad (L - \lambda_n r)\tilde{\theta}_o = r\tilde{\theta}_1 + \delta(t-y) ,$$

$$(2.23) \qquad (L - \lambda_n r)\tilde{\theta}_j = r\tilde{\theta}_{j+1} , \qquad j = 1,2,\ldots,(k-1) ,$$

with the boundary conditions

$$(2.24) \qquad \begin{aligned} \tilde{\theta}_{jt}(0,y,n) + \beta_o\tilde{\theta}_j(0,y,n) &= \alpha_o\lambda_n\tilde{\theta}_j(0,y,n) + \alpha_o\tilde{\theta}_{j+1}(0,y,n) ; \\ \tilde{\theta}_{jt}(1,y,n) + \beta_1\tilde{\theta}_j(1,y,n) &= \alpha_1\lambda_n\tilde{\theta}_j(1,y,n) + \alpha_1\tilde{\theta}_{j+1}(1,y,n) ; \end{aligned}$$

to be satisfied for

$$j = 0,1,2,\ldots,(k-1) .$$

The above equations can be satisfied only if the functions $\tilde{\theta}_j(t,y)$ satisfy the compatbility conditions

$$(2.25) \qquad Q[\psi_n(t),\tilde{\theta}_{j+1}(t,y,n)] = 0$$

for

$$j = 1,2,\ldots,(k-1) ;$$

and

$$(2.26) \qquad -Q[\psi_n(t),\tilde{\theta}_1(t,y,n)] = \psi_n(y) .$$

If $z = \lambda_n$ is a simple zero of $\omega(z)$ we find from (2.21) and (2.26) that

$$\tilde{\theta}_1(t,y,n) = \gamma_n\psi_n(t)$$

$$-Q[\psi_n(t),\gamma_n\psi_n(t)] = \psi_n(y)$$

$$\gamma_n = -\frac{\psi_n(y)}{Q[\psi_n(t),\psi_n(t)]}$$

and therefore (2.20) is

$$Q[\tilde{\theta}_1(t,y,n),f(t) = -\frac{Q[\psi_n(t),f(t)]\psi_n(y)}{Q[\psi_n(t),\psi_n(t)]} .$$

If each zero of $\omega(z)$ is simple, the expansion (2.19) becomes

$$f(y) = \sum_{n=0}^{\infty} \frac{Q[\psi_n(t),f(t)]\psi_n(y)}{Q[\psi_n(t),\psi_n(t)]} . \tag{2.27}$$

If $z = \lambda_n$ is a zero of $\omega(z)$ with multiplicity $k \neq 1$ we can see from (2.23) that $\tilde{\theta}_1(t,y,n)$ is not an eigenfunction but must satisfy

$$\{\frac{1}{r}(L - \lambda_n r)\}^{k-1}\,\tilde{\theta}_1(t,y,n) = \gamma_n\psi_n(t) . \tag{2.28}$$

For this case $\tilde{\theta}_1(t,y,n)$ is called a generalized eigenfunction.

We are now in a position to apply the foregoing results to the problem stated at the beginning of this section. For any finite x the function $\phi(x,y)$ can be expanded in the form

$$\phi(x,y) = -\sum_{n=0}^{\infty} Q[\tilde{\theta}_1(t,y,n),\phi(x,t)] . \tag{2.29}$$

The boundedness of $\phi(x,y)$ depends on the boundedness of each term

$$(2.30) \qquad \sigma_n(x) = Q[\tilde{\theta}_1(t,y,n),\phi(x,t)]$$

as a function of x. If we differentiate (2.30) with respect to x and express the derivative of the right-hand side in terms of $\tilde{\theta}_2$ using (2.23), we find

$$\sigma_n''(x) = -\int_0^1 \tilde{\theta}_1(t,y,n)L\phi \, dt$$

$$- p(1)\alpha_1\phi_{xx}(x,1)\tilde{\theta}_1(1,y,n)$$

$$+ p(0)\alpha_0\phi_{xx}(x,0)\tilde{\theta}_1(0,y,n)$$

which reduces to

$$\sigma_n''(x) + \lambda_n\sigma_n(x) = -Q[\tilde{\theta}_2(t,y,n),\phi(x,t)] \; .$$

We can see from the last result that if

$$D \equiv \frac{d}{dx}$$

then $\sigma_n(x)$ must satisfy

$$(2.31) \qquad [D^2 + \lambda_n]^k \, \sigma_n(x) = 0$$

where k is the order of the pole of $G(t,y,n)$ at $z = \lambda_n$. The general solution of (2.31) is

$$(2.32) \qquad \sigma_n(x) = e^{ix\sqrt{\lambda_n}} \sum_{j=0}^{k-1} a_{nj} x^j + e^{-ix\sqrt{\lambda_n}} \sum_{j=0}^{k-1} b_{nj} x^j \ .$$

Therefore if $\sigma_n(x)$ is to be bounded for all x we must choose

$$a_{nj} = b_{nj} = 0 \ , \quad j = 1,2,\ldots,(k-1) \ .$$

Then what remains, namely

$$\sigma_n(x) = a_{no} e^{ix\sqrt{\lambda_n}} + b_{no} e^{-ix\sqrt{\lambda_n}} \ ,$$

is bounded if λ_n is a real non-negative eigenvalue. If λ_n does not satisfy this condition then we must also take

$$a_{no} = b_{no} = 0$$

in order to have $\sigma_n(x)$ bounded.

We conclude that the answer to the question about the existence of bounded solutions of (2.1)-(2.3), other than $\phi(x) = 0$, depends on the disposition of the eigenvalues of (2.4)-2.6). For example, if the parameters of (2.4)-(2.6) are such that all of the eigenvalues are negative then the only bounded solution of (2.1)-(2.4) is $\phi(x) = 0$.

The method explained in this section can be applied to the two-dimensional linear analysis of the flow of a gravitating incompressible and inviscid liquid which is confined to an

infinitely long open channel with a horizontal bottom and vertical walls. Let the density ρ of the liquid be constant; let the only body force be the gravitational force ρg acting in the direction of the negative y-axis and let the x-axis coincide with the bottom of the channel. The basic hydrodynamical equations can be written in the dimensionless form

$$\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0 \ , \tag{2.33}$$

$$\begin{cases} \dfrac{\partial u}{\partial \tau} + u_1 \dfrac{\partial u_1}{\partial x} + u_2 \dfrac{\partial u_1}{\partial y} = - \dfrac{2\pi_1}{\partial x} \ , \\[2ex] \dfrac{\partial u}{\partial \tau} + u_1 \dfrac{\partial u_2}{\partial x} + u_2 \dfrac{\partial u_2}{\partial y} = - 1 - \dfrac{2\pi_1}{\partial y} \ . \end{cases} \tag{2.34}$$

Here u_1 and u_2 correspond respectively to the horizontal and vertical components of velocity; while τ and π_1 are respectively proportional to the time and pressure. The boundary conditions which must be satisfied at the free surface

$$y = f_1(x,\tau) \ ,$$

where we take the pressure to be zero, are

$$u_2 = u_1 \frac{\partial f_1}{\partial x} + \frac{\partial f_1}{\partial \tau} \tag{2.35}$$

and

$$\pi_1(x,f_1,\tau) = 0 \ . \tag{2.36}$$

The boundary condition for the bottom is

$$u_2(x.0) = 0 \ . \tag{2.37}$$

This set of partial differential equations and boundary conditions are satisfied by the quantities

$$\left\{\begin{aligned} & u_1 = u_o(y) = \frac{c + v_o(y)}{\sqrt{gh}} \ , \\ & u_2 = 0 \ , \quad f_1 = 1 \ , \\ & \pi_1 = 1 - y \ , \end{aligned}\right. \tag{2.38}$$

where c is a constant and $v_o(y)$ is a continuous non-negative function. These quantities define a steady parallel motion in the channel with the dimensionless depth of the liquid equal to unity; and we will refer to this flow as the equilibrium flow. The velocity $v_o(y)$ gives the shear in the axial velocity component and it is also a measure of the departure of the flow from a uniform state defined by the velocity c.

Let us proceed to linearize the equations (2.33)-(2.37) with respect to the flow defined by (2.38). That is, let us write

$$\left\{\begin{aligned} & u_1 = u_o(y) + u(x,y) \\ & u_2 = v(x,y) \\ & \pi_1 = 1 - y + \phi(x,y) \\ & f_1 = 1 + f(x) \end{aligned}\right. \tag{2.39}$$

and assume that each of u, v, ϕ and f is independent of the time and small with a common order of magnitude. Let us substitute these quantities in (2.33)-(2.37) and retain only first order terms. The result of the linearization of equations (2.33)-(2.34) is

$$u_x + v_y = 0 , \tag{2.40}$$

$$u_o u_x + u_{oy} v = - \phi_x , \tag{2.41}$$

$$u_o v_x = - \phi_y . \tag{2.42}$$

The condition to be satisfied at the bottom of the channel is

$$v(x,0) = 0 . \tag{2.43}$$

Subject to the linearization, the free surface conditions become conditions to be satisfied at y = 1. In place of (2.35) we have

$$v(x,1) = u_o(1) f_x(1,x) , \tag{2.44}$$

and from (2.36) we have

$$- f + \phi(x, 1 + f) = 0$$

which after differentiation and removal of higher order terms becomes

$$- f_x(x) + \phi_x(x,1) = 0 . \tag{2.45}$$

The elimination of u, v and f from the linear equations (2.40)-(2.45) shows that $\phi(x,y)$ must satisfy

(2.46) $$\frac{\partial}{\partial y}\frac{1}{u_o^2(y)}\frac{\partial}{\partial y}\phi(x,y)+\frac{1}{u_o^2(y)}\frac{\partial^2\phi}{\partial x^2}=0\ , \qquad -\infty < x < \infty\ , \quad 0 < y < 1$$

subject to the boundary conditions

(2.47) $$\phi_y(x,1)+u_o^2(1)\phi_{xx}(x,1)=0\ ,$$

and

(2.48) $$\phi_y(x,0)=0\ .$$

These equations, (2.46)-(2.48), constitute a particular case of the system (2.1)-(2.3) which we have already analyzed.

The system (2.46)-(2.48) is satisfied by $\phi(x,y) = \text{const.}$ This, according to the linear theory, is a necessary and sufficient condition for the existence of an equilibrium flow in the channel. We now ask: What conditions must $u_o(y)$ satisfy in order to insure that $\phi = \text{const.}$ is the only bounded solution of (2.46)-(2.48)? As we have seen, the answer to this question can be made to depend on the disposition of the eigenvalues of

(2.49) $$\frac{d}{dy}\frac{1}{u_o^2(y)}\frac{d}{dy}\psi(y)-\frac{\lambda}{u_o^2(y)}\psi=0\ , \qquad 0 \le y \le 1\ ,$$

(2.50) $$\psi_y(1)=u_o^2(1)\lambda\psi(1)\ ,$$

(2.51) $$\psi_y(0)=0\ .$$

It is easy to see that all of the eigenvalues of (2.49)-(2.51) are real and that $\lambda_o = 0$ is always an eigenvalue for which we can take $\psi_o = 1$ as the corresponding eigenfunction. It is also evident that an eigenfunction corresponding to a positive eigenvalue cannot vanish at the end points $y = 0, 1$ nor can it vanish at $y = \eta$ where $0 < y = \eta < 1$. If it did, that is, if $\psi(\eta) = 0$ we would have

$$\int_0^\eta \psi(y) \frac{d}{dy} \frac{1}{u_o^2(y)} \frac{d}{dy} \psi dy = \lambda \int_0^\eta \frac{\psi^2(y)}{u_o^2(y)} dy$$

which after integration and use of the boundary conditions gives

$$-\int_0^\eta \frac{[\psi_y(y)]^2}{u_o^2(y)} dy = \lambda \int_0^\eta \frac{\psi^2(y)}{u_o^2(y)} dy ,$$

a contradiction if λ is positive. (All of the eigenfunctions, except for a multiplicative factor, must be real.) It follows that if λ is positive, we can divide (2.49) by ψ and integrate so as to get

$$\int_0^1 \frac{1}{\psi} \frac{d}{dy} \frac{1}{u_o^2} \frac{d}{dy} \psi(y) dy = \lambda \int_0^1 \frac{dy}{u_o^2} ,$$

$$\lambda + \int_0^1 \frac{\psi_y^2}{u_o^2 \psi^2} dy = \lambda \int_0^1 \frac{dy}{u_o^2} ,$$

$$\lambda \left[\int_0^1 \frac{dy}{u_o^2} - 1 \right] = \int_0^1 \frac{\psi_y^2}{u_o^2 \psi^2} dy$$

which shows that a positive eigenvalue cannot exist if

$$\int_0^1 \frac{dy}{u_o^2} < 1 \ . \tag{2.52}$$

Furthermore, if we integrate (2.49) we find

$$\frac{1}{u_o^2(y)} \psi_y(y) \Big|_0^1 - \lambda \int_0^1 \frac{\psi}{u_o^2} dy = 0$$

or

$$\lambda \left[\psi(1) - \int_0^1 \frac{\psi}{u_o^2} dy \right] = 0 \ . \tag{2.53}$$

This can be regarded as an equation for the determination of the eigenvalues of (2.49)-(2.51). If we expand ψ into a power series in λ and substitute in (2.53) we find that

$$\int_0^1 \frac{dy}{u_o^2} = 1 \tag{2.54}$$

is the condition for a multiple eigenvalue at $\lambda = 0$. That is, if

$$\int_0^1 \frac{dy}{u_o^2} - 1$$

is negative and then made to increase to zero by changing u_o, the negative eigenvalue of least absolute magnitude must approach zero.

It follows from the above remarks and the criterion deduced from (2.32) that ϕ = const. is the only bounded solution of (2.46)-(2.48) if

$$u_o = \frac{c + v_o(y)}{\sqrt{gh}}$$

satisfies

$$gh \int_0^1 \frac{dy}{[c + v_o(y)]^2} \leq 1 \ . \tag{2.55}$$

If $v_o(y) = 0$ the condition (2.55) shows that the linear hydrodynamical theory predicts that the uniform flow defined by

$$u = \frac{c}{\sqrt{gh}}$$

$$v = 0$$

is a unique bounded flow if the speed c is not less than the critical speed $\sqrt{gh}$ where g is the acceleration due to gravity and h is the depth of the liquid.

Another analysis of the foregoing channel problem and the problem for a liquid with non-constant density can be found in a report by Peters [9] which contains a more detailed discussion of (2.55) and other results.

3. Churchill's Method

If certain conditions are satisfied, the eigenvalue problem (2.4)-(2.6), namely

$$\frac{d}{dy}\, p(y)\, \frac{d}{dy}\, \psi(y) + q(y)\psi - \lambda r(y)\psi = 0 \ , \qquad 0 < y < 1 \ , \tag{3.1}$$

$$(3.2) \qquad \psi_y(0) + \beta_0\psi(0) = \alpha_0\lambda\psi(0) ,$$

$$(3.3) \qquad \psi_y(1) + \beta_1\psi(1) = \alpha_1\lambda\psi(1)$$

can be reduced to an ordinary Sturm-Liouville problem by using an appropriate substitution for $\psi(y)$. For the case where the α's and β's are real with $\alpha_1 \leq 0$ while $\alpha_0 \geq 0$, Churchill devised a procedure for reducing (3.1)-(3.3) to an eigenvalue problem in which the eigenparameter does not appear in the boundary conditions. We proceed to show that with α_0 and α_1 subject to no restrictions with respect to sign, Churchill's method can still be used; provided there exists a positive function $\psi_0(y)$ which satisfies

$$(3.4) \qquad \frac{d}{dy}\, p(y)\, \frac{d}{dy}\, \psi_0(y) + q\psi_0 - \lambda_0 r\psi_0 = 0 , \qquad 0 \leq y \leq 1 ,$$

$$(3.5) \qquad \psi_0'(0) + \beta_0\psi(0) = \alpha_0\lambda_0\psi_0(0) ,$$

$$(3.6) \qquad \psi_0'(1) + \beta_1\psi(1) = \alpha_1\lambda_0\psi_0(1) ,$$

and

$$(3.7) \qquad \psi_0(y) > 0 , \qquad 0 \leq y \leq 1 .$$

Suppose that λ_0 is real and that it is the least non-negative value corresponding to which such a $\psi_0(y)$ exists. Define $w(y)$ by

$$(3.8) \qquad \psi(y) = \psi_0(y)\left[\int_0^y \frac{w(\eta)d\eta}{\psi_0^2(\eta)p(\eta)} + a\right] .$$

Under this substitution, with the eigenvalue problem (3.1)-(3.3), with $\lambda \neq \lambda_o$, is changed into the one defined by

$$\frac{d}{dy}\frac{1}{r\psi_o^2}\frac{dw}{dy} - (\lambda - \lambda_o)\frac{w}{p\psi_o^2} = 0 \ , \tag{3.9}$$

$$r(0)w(0) = \alpha_o p(0)w'(0) \ , \tag{3.10}$$

$$r(1)w(1) = \alpha_1 p(1)w'(1) \ . \tag{3.11}$$

Conversely, this standard Sturm-Liouville problem is transformed by (3.8) into the problem of finding ψ such that

$$\frac{d}{dy}\, p\psi' + q\psi - \lambda r\psi = r\psi_o \left[\frac{w'(0)}{r(0)\psi_o^2(0)} - (\lambda - \lambda_o)a\right] \ , \tag{3.12}$$

$$\psi'(0) + \beta_o\psi(0) = \alpha_o\lambda\psi(0) + \psi_o(0)\left[\frac{w'(0)}{r(0)\psi_o^2(0)} - (\lambda - \lambda_o)a\right] \ , \tag{3.13}$$

and

$$\psi'(1) + \beta_1\psi(1) = \alpha_1\lambda\psi(1) + \psi_o(1)\left[\frac{w'(0)}{r(0)\psi_o^2(0)} - (\lambda - \lambda_o)a\right] \ . \tag{3.14}$$

If $\lambda = \lambda_o$ is not an eigenvalue of (3.9)-(3.11) the constant a can be fixed by

$$\frac{w'(0)}{r(0)\psi_o^2(0)} - (\lambda - \lambda_o)a = 0 \tag{3.15}$$

and then (3.9)-(3.11) is equivalent to (3.1)-(3.3).

Suppose next that $\lambda = \lambda_o$ is an eigenvalue of (3.9)-(3.11) and that the corresponding eigenfunction is $w_o(y)$ so that

$$(3.16) \qquad \frac{d}{dy} \frac{1}{r\psi_o^2} \frac{dw_o}{dy} = 0 ,$$

$$(3.17) \qquad r(0)w_o(0) = \alpha_o p(0)w_o'(0) ,$$

$$(3.18) \qquad r(1)w_o(1) = \alpha_1 p(1)w_o'(1) .$$

The function

$$(3.19) \qquad w_o(y) = k\left[\int_0^y r(\eta)\psi_o^2(\eta)d\eta + \alpha_o p(0)\psi_o^2(0)\right]$$

$$k = \frac{w_o'(0)}{r(0)\psi_o^2(0)}$$

satisfies (3.16) and (3.17). The condition (3.18) then shows that λ_o is an eigenvalue if and only if

$$(3.20) \qquad \int_0^1 r(y)\psi_o^2(y)dy - \alpha_1 p(1)\psi_o^2(1) + \alpha_o p(0)\psi_o^2(0) = 0$$

or

$$(3.21) \qquad Q[\psi_o(y),\psi_o(y)] = 0$$

if we use the symbol defined in Section 2. The function $\tilde{\psi}(y)$ which corresponds to w_o is, in accordance with (3.8), defined by

$$\frac{d}{dy}\frac{\tilde{\psi}(y)}{\psi_o(y)} = \frac{w_o(y)}{p(y)\psi_o^2(y)} \ ;$$

and it is easy to verify that $\tilde{\psi}$ must satisfy

$$\frac{d}{dy}\, p(y)\, \frac{d}{dy}\, \tilde{\psi}(y) + q\tilde{\psi} - \lambda_o r\tilde{\psi} = kr\psi_o \ , \tag{3.22}$$

$$\tilde{\psi}'(0) + \beta_o\tilde{\psi}(0) = \alpha_o\lambda_o\tilde{\psi}(0) + \alpha_o k\psi_o(0) \ , \tag{3.23}$$

$$\tilde{\psi}'(1) + \beta_1\tilde{\psi}(1) = \alpha_1\lambda_o\tilde{\psi}(1) + \alpha_1 k\psi_o(1) \ , \tag{3.24}$$

where $k \neq 0$. The function $\tilde{\psi}(y)$ is not an eigenfunction of (3.1)-(3.3). However, if the set of eigenfunctions of (3.1)-(3.3) is extended by the addition of $\tilde{\psi}(y)$ then, as we shall see, an arbitrary function can be expanded with respect to the enlarged set.

Under our assumptions, the eigenvalues of (3.9)-(3.11) are all real. These eigenvalues can be ordered with respect to absolute magnitude, that is,

$$|\lambda_o| \leq |\lambda_1| \leq \cdots \ |\lambda_n| \leq \cdots \ ;$$

and it is well known that the corresponding set of eigenfunctions is complete. If

$$Q[\psi_o(y),\psi_o(y)] \neq 0 \ , \tag{3.25}$$

the set of eigenfunctions can be designated by

$$\{w_n(y)\} \qquad n = 1,2,3,\ldots .$$

The eigenfunctions satisfy

$$\int_0^1 \frac{w_n(y)w_m(y)}{p(y)\psi_o^2(y)}\,dy = 0 , \qquad m \neq n .$$

If $F(y)$ is differentiable we have

$$F(y) = \sum_{n=1}^{\infty} \frac{\displaystyle\int_0^1 \frac{F(t)w_n(t)}{p(t)\psi_o^2(t)}\,dt}{\displaystyle\int_0^1 \frac{w_n^2(t)}{p(t)\psi_o^2(t)}\,dt}\, w_n(y) .$$

Hence if $f(y)$ is an arbitrary differentiable function we have

(3.26) $$\int_0^y \frac{d}{dt}\left|\frac{f(t)}{\psi_o(t)}\right| dt = \int_0^y \frac{1}{p(t)\psi_o^2(t)}\left\{\sum_{n=1}^{\infty} \gamma_n w_n(t)\right\} dt$$

where

(3.27) $$\gamma_n = \frac{\displaystyle\int_0^1 w_n(t)\,\frac{d}{dt}\left(\frac{f}{\psi_o}\right)dt}{\displaystyle\int_0^1 \frac{w_n^2(t)}{p\psi_o^2}\,dt} , \qquad n = 1,2,\ldots .$$

In terms of the set, $\{\psi_n(y)\}$, of eigenfunctions of (3.1)-(3.3), including $\psi_o(y)$, the expansion (3.26) implies

(3.28) $$f(y) = \sum_{n=0}^{\infty} s_n\psi_n(y)$$

where

$$(3.29) \qquad s_n = \frac{Q[f,\psi_n]}{Q[\psi_n,\psi_n]} .$$

If

$$(3.30) \qquad Q[\psi_o(t),\psi_o(t)] = \int_0^1 r(t)\psi_o^2(t)dt - \alpha_1 p(1)\psi_o^2(1) + \alpha_o p(0)\psi_o^2(0)$$

$$= 0$$

the eigenfunctions of (3.9)-(3.11) can be designated by

$$\{w_n(y)\} \qquad n = 0,1,2,\dots .$$

For this case we have

$$(3.31) \qquad \int_0^y \frac{d}{dt}\left|\frac{f(t)}{\psi_o(t)}\right|dt = \int_0^y \frac{1}{p(t)\psi_o^2(t)}\left\{\sum_{n=0}^{\infty} \gamma_n w_n(t)\right\} dt$$

where

$$(3.32) \qquad \gamma_n = \frac{\displaystyle\int_0^1 w_n(t)\frac{d}{dt}\left(\frac{f}{\psi_o}\right)dt}{\displaystyle\int_0^1 \frac{w_n^2(t)}{p(t)\psi_o^2(t)}dt}, \qquad n = 0,1,2,\dots .$$

This can be expressed in terms of the eigenfunctions, $\{\psi_n(y)\}$, of (3.1)-(3.3) and the generalized eigenfunction $\tilde{\psi}(y)$. A calculation shows that

$$(3.33) \qquad f(y) = s\tilde{\psi}(y) + \tilde{s}\psi_o(y) + \sum_{n=1}^{\infty} s_n\psi_n(y)$$

where

(3.34) $$s_n = \frac{Q[f,\psi_n]}{Q[\psi_n,\psi_n]} \quad ,$$

(3.35) $$s = \frac{Q[f,\psi_o]}{Q[\psi_o,\tilde{\psi}]} \quad ;$$

and

(3.36) $$\tilde{s} = \frac{Q[f,\tilde{\psi}]Q[\psi_o,\tilde{\psi}] - Q[f,\psi_o]Q[\tilde{\psi},\tilde{\psi}]}{\{Q[\psi_o,\tilde{\psi}]\}^2} \quad .$$

The expansions (3.28) and (3.33) are particular cases of the basic general expansion (2.19). It appears that the assumption with respect to the existence of $\psi_o(y)$ implies that, excepting a possible multiple pole at $z = \lambda_o$, $G(t,y,z)$ possesses only simple poles.

When $\psi_o(y)$ exists the question of the existence of a non-trivial bounded solution of (2.1)-(2.3) reduces to finding the disposition of the eigenvalues of (3.9)-(3.11). Under certain circumstances this disposition can be exposed more clearly by integrating (3.9). When $\lambda \neq \lambda_o$ integration of (3.9) namely,

$$\frac{d}{dy}\frac{1}{r\psi_o^2}\frac{dw}{dy} - (\lambda - \lambda_o)\frac{w}{p\psi_o^2} = \frac{d}{dy}\left\{\frac{1}{r\psi_o^2}\frac{dw}{dy} - (\lambda - \lambda_o)\frac{\psi}{\psi_o}\right\}$$

gives

$$\frac{1}{r\psi_o^2}\frac{dw}{dy} - (\lambda - \lambda_o)\frac{\psi}{\psi_o} = 0$$

or

(3.37) $$\frac{1}{r\psi_o^2}\frac{d}{dy}p\psi_o^2\frac{d}{dy}\left(\frac{\psi}{\psi_o}\right) = (\lambda - \lambda_o)\frac{\psi}{\psi_o} \quad .$$

Suppose now that if λ is a positive eigenvalue such that $\lambda > \lambda_o$ then the corresponding eigenfunction does not vanish in the interval $0 \leq y \leq 1$. When this condition prevails we can multiply (3.37) by $r\psi_o^3/\psi$ to obtain

$$\frac{\psi_o}{\psi} \frac{d}{dy} p\psi_o^2 \frac{d}{dy} \left(\frac{\psi}{\psi_o}\right) = (\lambda - \lambda_o) r\psi_o^2 .$$

Another integration here, produces

$$(3.38) \quad \int_0^1 \left(\frac{\psi_o}{\psi}\right)^2 \left[\frac{d}{dy} \left(\frac{\psi}{\psi_o}\right)\right]^2 p\psi_o^2 dy$$

$$= (\lambda - \lambda_o) \left\{ \int_0^1 r\psi_o^2 dy - \alpha_1 p(1)\psi_o^2(1) + \alpha_o p(0)\psi_o^2(0) \right\} .$$

This shows that, subject to the supposition above, there cannot be a real eigenvalue λ such that $\lambda > \lambda_o$ if

$$Q[\psi_o, \psi_o] = \int_0^1 r\psi_o^2 dy - \alpha_1 p(1)\psi_o^2(1) + \alpha_o p(0)\psi_o^2(0)$$

is negative. When this criterion is applied to (2.49)-(2.51) it yields (2.52).

4. Application of a Transform Method

The system which was analyzed in Section 2, namely

$$(4.1) \qquad \frac{\partial}{\partial y} p(y) \frac{\partial}{\partial y} \phi(x,y) + q(y)\phi + r(y) \frac{\partial^2 \phi}{\partial x^2} = 0 , \qquad -\infty < x < \infty , \quad 0 < y < 1 ,$$

$$(4.2) \qquad \phi_y(x,0) + \alpha_o \phi_{xx}(x,0) + \beta_o \phi(x,0) = 0 ,$$

$$(4.3) \qquad \phi_y(x,1) + \alpha_1 \phi_{xx}(x,1) + \beta_1 \phi(x,1) = 0 ;$$

can also be studied by using a generalized Fourier transform method.

Let the right-hand Fourier transform of $\phi(x,y)$ be

$$\Phi(\lambda,y) = \int_0^\infty e^{i\lambda x} \phi(x,y)dx$$

where Im $\lambda = a > 0$. Let the left-hand transform of ϕ be

$$\Phi_1(\lambda,y) = \int_{-\infty}^0 e^{i\lambda x} \phi(x,y)dx$$

where Im $\lambda = b < 0$. By taking the magnitudes of a and b sufficiently large these transforms exist for a twice differentiable function ϕ of any exponential order. The recovery formula for $\phi(x,y)$ is

$$\phi(x,y) = \frac{1}{2\pi} \int_{-\infty+ia}^{\infty+ia} e^{-ix\lambda} \Phi \, d\lambda + \frac{1}{2\pi} \int_{-\infty+ib}^{\infty+ib} e^{-ix\lambda} \Phi_1 d\lambda .$$

If ϕ and its first derivative with respect to x are of exponential order, the application of the right-hand transform to (4.1)-(4.3) for $x > 0$ shows that Φ must satisfy

$$(4.4) \qquad \frac{\partial}{\partial y} p(y)\Phi_y(\lambda,y) + q\Phi - \lambda^2 r\Phi = r[\phi_x(0,y) - i\lambda\phi(0,y)]$$

for $0 < y < 1$ subject to the boundary conditions

$$(4.5) \qquad \Phi_y(\lambda,0) + \beta_o\Phi(\lambda,0) = \alpha_o\lambda^2\Phi(\lambda,0) + \alpha_o[\phi_x(0,0) - i\lambda\phi(0,0)]$$

and

$$(4.6) \qquad \Phi_y(\lambda,1) + \beta_1\Phi(\lambda,1) = \alpha_1\lambda^2\Phi(\lambda,1) + \alpha_1[\phi_x(0,1) - i\lambda\phi(0,1)] \ .$$

Similarly, the application of the left-hand transform to (4.1)-(4.3) for $x < 0$ shows that Φ_1 must satisfy

$$(4.7) \qquad \frac{\partial}{\partial y} p\Phi_{1y} + q\Phi_1 - \lambda^2 r\Phi_1 = -r[\phi_x(0,y) - i\lambda\phi(0,y)]$$

for $0 < y < 1$ with the boundary conditions

$$(4.8) \qquad \Phi_{1y}(\lambda,0) + \beta_o\Phi_1(\lambda,0) = \alpha_o\lambda^2\Phi_1(\lambda,0) - \alpha_o[\phi_x(0,0) - i\lambda\phi(0,0)] \ ,$$

$$(4.9) \qquad \Phi_{1y}(\lambda,1) + \beta_1\Phi_1(\lambda,0) = \alpha_1\lambda^2\Phi_1(\lambda,1) - \alpha_1[\phi_x(0,1) - i\lambda\phi(0,1)] \ .$$

From the equations and boundary conditions which Φ and Φ_1 must satisfy it is evident that

$$\Phi_1(\lambda,y) = \Phi(\lambda,y) \ .$$

Therefore we see that

$$(4.10) \qquad \phi(x,y) = -\frac{1}{2\pi}\int_M e^{-ix\lambda}\,\underline{\Phi}(\lambda,y)\,d\lambda$$

where M is the path $M = M_1 + M_2$ shown in Fig. 4.1. The lines M_1 and

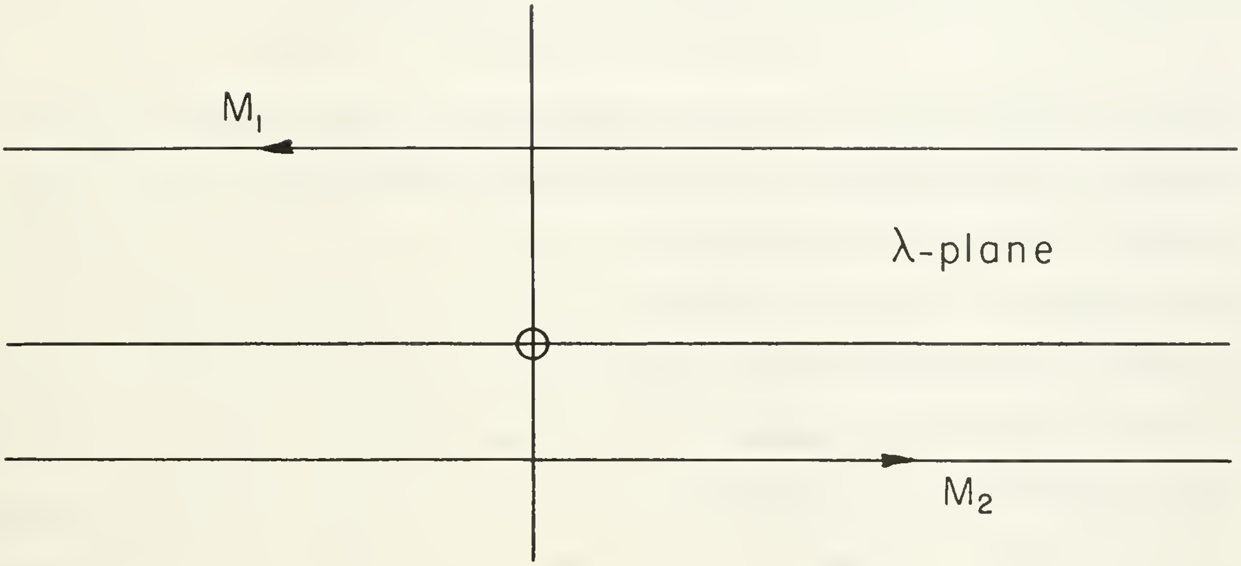

Fig. 4.1

M_2 are parallel to the real axis of the λ-plane and their respective distances a and $|b|$ from the real axis can be adjusted to admit functions ϕ of various exponential orders. Also, by proper choice of a and b we can allow the exponential order of behavior of ϕ as $x \longrightarrow \infty$ to be different from that as $x \longrightarrow -\infty$.

It can be shown that $\underline{\Phi}$ is expressible as a ratio

$$\underline{\Phi}(\lambda,y) = \frac{\psi(y,\lambda)}{\omega(\lambda)}$$

in which each of ψ and ω is an entire function of the complex

variable λ. The zeros of $\omega(\lambda)$ are just the eigenvalues of

$$\frac{d}{dy} p\psi_y(y,\lambda) + q\psi - \lambda^2 r\psi = 0 \ , \qquad 0 < y < 1 \ , \tag{4.11}$$

$$\psi_y(0,\lambda) + \beta_o\psi(0,\lambda) = \alpha_o\lambda^2\psi(0,\lambda) \ , \tag{4.12}$$

$$\psi_y(1,\lambda) + \beta_1\psi(1,\lambda) = \alpha_1\lambda^2\psi(1,\lambda) \ . \tag{4.13}$$

If the disposition of these eigenvalues is known and if the behavior of $\psi(y,\lambda)/\omega(\lambda)$ as $\lambda \longrightarrow \infty$ can be estimated, then the behavior of $\phi(x,y)$ with respect to x can be found from (4.10) by using the theory of residues. Hence our problem is again reduced to a study of the eigenfunction system which was introduced in Section 2.

Suppose for example that the parameters of (4.11)-(4.13) are such that all of the eigenvalues are pure imaginaries <u>not including</u> $\lambda = 0$. For a bounded solution ϕ we can take $a = |b| = \varepsilon$ so small that M does not contain any of these eigenvalues. Then, since our assumptions on the order of ϕ and its first derivative with respect to x imply

$$\lim_{N \longrightarrow \pm\infty} \int_{-\varepsilon}^{\varepsilon} e^{-i(N+i\eta)x} \, \underline{\Phi}(N+i\eta,y)d\eta = 0 \ ,$$

it follows that

$$\phi(x,y) = -\frac{1}{2\pi}\int_M e^{-ix\lambda} \, \underline{\Phi}(\lambda,y)d\lambda = 0$$

is the only bounded solution of (4.1)-(4.3) if (4.11)-(4.13) possesses no real eigenvalues.

It should be noted that the above transform method does not require a knowledge of the completeness or lack of completeness of the eigenfunctions of (4.11)-(4.13). Hence the infinite transform method appears to be a direct one for at least the investigation of bounded solutions of (4.1)-(4.3). However, this method starts with assumptions about the behavior of ϕ and its derivatives at infinity whereas the generalized eigenfunction methods of Sections 2 and 3 do not require such initial assumptions. Furthermore, the transform method depends on an estimate of the transform $\underline{\Phi}(\lambda,y)$ as $\lambda \longrightarrow \infty$. A consideration of these facts and a comparison of a transform method with the method of Section 2 as these methods are conceptually applied to problems involving domains more general than the doubly infinite strip domain, leads to the conclusion that the eigenfunction method is more fundamental. For example, the eigenfunction method can be applied to problems involving a rectangular domain for which a finite transform method is not directly effective because an inversion formula for the transform is not at hand and needs to be derived. The derivation of such a formula is in general equivalent to establishing an eigenfunction expansion.

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DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Courant Institute of Mathematical Sciences New York University	2a. REPORT SECURITY CLASSIFICATION not classified 2b. GROUP none

3. REPORT TITLE

Some Generalized Eigenfunction Expansions and Uniqueness Theorems

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Technical Report June 1968

5. AUTHOR(S) (Last name, first name, initial)

Peters, Arthur S.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
June 1968	39	10

8a. CONTRACT OR GRANT NO. Nonr-285(55) b. PROJECT NO. NR 062-160 c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) IMM 368 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) none

10. AVAILABILITY/LIMITATION NOTICES

Distribution of this document is unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
none	U.S. Navy, Office of Naval Research 207 West 24th St., New York, N.Y.

13. ABSTRACT A generalized eigenfunction expansion method, Churchill's method, and a transform method are used to investigate the uniqueness of the solution of the equation

$$\frac{\partial}{\partial y} p(y) \frac{\partial}{\partial y} \phi(x,y) + q(y)\phi + r(y) \frac{\partial^2 \phi}{\partial x^2} = 0 , \qquad -\infty < x < \infty , \quad 0 < y < 1$$

subject to the boundary conditions

$$\phi_y(x,0) + \alpha_o \phi_{xx}(x,0) + \beta_o \phi(x,0) = 0$$

and

$$\phi_y(x,1) + \alpha_1 \phi_{xx}(x,1) + \beta_1 \phi(x,1) = 0 .$$

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